

Semiclassical Density of States for the Quantum Asymmetric Top

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Abstract

In the quantization of a rotating rigid body, a *top*, one is concerned with the Hamiltonian operator $L_\alpha = \alpha_0^2 L_x^2 + \alpha_1^2 L_y^2 + \alpha_2^2 L_z^2$, where $\alpha_0 < \alpha_1 < \alpha_2$. An explicit formula is known for the eigenvalues of L_α in the case of the spherical top ($\alpha_1 = \alpha_2 = \alpha_3$) and symmetrical top ($\alpha_1 = \alpha_2 \neq \alpha_3$) [LL]. However, for the asymmetrical top, no such explicit expression exists, and the study of the spectrum is much more complex. In this paper, we compute the semiclassical density of states for the eigenvalues of the family of operators $L_\alpha = \alpha_0^2 L_x^2 + \alpha_1^2 L_y^2 + \alpha_2^2 L_z^2$ for any $\alpha_0 < \alpha_1 < \alpha_2$.

1 Introduction

Let $S^2 \subseteq \mathbb{R}^3$ be the 2-sphere and let $-\Delta_{S^2}$ be the constant curvature spherical Laplacian on S^2 . It is well known that the spectrum of $-\Delta_{S^2}$ consists of eigenvalues λ given by

$$\lambda_k = k(k+1), \quad k = 0, 1, 2, \dots$$

Moreover, the eigenspace corresponding to λ_k is of dimension $2k+1$ and a basis of eigenfunctions is obtained by taking the standard spherical harmonics of degree k , i.e.

$$Y_k^m(\theta, \phi) = P_k^m(\cos \theta) e^{im\phi}, \quad |m| \leq k,$$

where P_k^m is the associated Legendre function of the first kind. For a more detailed treatment of the spectral theory of Δ_{S^2} , we refer the reader to [Fo].

From the fact that the eigenvalues $\lambda_k = k(k+1)$ of $-\Delta_{S^2}$ are of multiplicity $2k+1$, it is easy to see that spectrum of $-\Delta_{S^2}$ has clustering. A nice way to illustrate this fact is to observe that for any Schwartz function φ on \mathbb{R} ,

$$\frac{1}{2k+1} \sum_{j=-k}^k \varphi\left(\frac{\sqrt{\lambda_k}}{k}\right) = \phi(1) + \mathcal{O}\left(\frac{1}{k}\right), \quad (1.1)$$

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in the semi-classical limit $k \rightarrow \infty$ (see [M]). Expressions like those appearing on the RHS of (1.1) are often referred to as a density of states (DOS) (see e.g. [T1]). Together with the mean level spacings and the pairs correlation, the DOS represents a useful quantity to measure the spread of the spectrum.

In this paper we are interested in computing the DOS for $\sqrt{-L_\alpha}$, associated with the quantum asymmetric top with Hamiltonian L_α , where L_α is given by

$$L_\alpha := (\alpha_0^2 L_x^2 + \alpha_1^2 L_y^2 + \alpha_2^2 L_z^2),$$

and where

$$L_x = -i(y\partial_z - z\partial_y),$$

$$L_y = -i(z\partial_x - x\partial_z),$$

$$L_z = -i(x\partial_y - y\partial_x).$$

Here, we assume that $\alpha = (\alpha_0^2, \alpha_1^2, \alpha_2^2) \in \Lambda^3$, where

$$\Lambda^3 := \left\{ \alpha \in \mathbb{R}^3 : 0 < \alpha_0^2 < \alpha_1^2 < \alpha_2^2 \right\}$$

is the positive Weyl chamber. It is well known that $-\Delta_{S^2}$ and $-L_\alpha$ are commuting, self-adjoint, elliptic operators on $L^2(S^2)$ and therefore possess a Hilbert basis of joint eigenfunctions – the aforementioned spherical harmonics Y_m^k [BT]. Moreover, it is easy to verify that their principal symbols are linearly independent in $T^*(S^2)$. For these reasons, we say that Δ_{S^2} and L_α form a quantum integrable system on S^2 .

An explicit formula is known for the eigenvalues of L_α in the case of the spherical top ($\alpha_1 = \alpha_2 = \alpha_3$) and symmetrical top ($\alpha_1 = \alpha_2 \neq \alpha_3$) [LL]. Although no such explicit formula exists for the eigenvalues of the asymmetrical top ($\alpha_1 \neq \alpha_2 \neq \alpha_3$), the spectrum was recently characterized in terms of parameters associated with the Lamé equation (cf. proposition 2.2 in [T2]).

For such a system, it is customary to compute the DOS of their joint spectrum (see e.g. [Ch, Co]). Here, we are simply concerned with the density of states measures associated to the operators $\sqrt{-L_\alpha}$. In the following, we denote by E_k the eigenspace of Δ_{S^2} consisting of spherical harmonics of degree k , i.e. $E_k = \text{Span}\{Y_m^k : m = -k, -k+1, \dots, k\}$, and by P_k the projection onto E_k . We define the DOS measure associated to the operators L_α^2 by

$$d\rho_{DS}(x; k, \alpha) := \frac{1}{2k+1} \sum_{\lambda \in \sigma(\sqrt{-P_k L_\alpha})} \delta\left(x - \frac{\lambda}{k}\right) \quad (1.2)$$

where $\sigma(\sqrt{-P_k L_\alpha})$ denotes the spectrum of $\sqrt{-P_k L_\alpha}$. Clearly, $\sigma(\sqrt{-P_k L_\alpha})$ consists of the eigenvalues $\sqrt{\lambda_m^k}$, $m \leq |k|$, of $\sqrt{-L_\alpha}$ associated to the spherical harmonics of degree k . Our purpose here is to compute the density of states for the measure $d\rho_{DS}(x, k; \alpha)$ in the semi-classical regime $k \rightarrow \infty$.

1.1 Main result

For any given $\alpha \in \Lambda^3$, let g be the function defined on the rectangle $[0, \pi] \times [0, \pi/2]$ by

$$g(\xi, \theta; \alpha) = (\alpha_1^2 - \alpha_0^2) (\beta \cos \xi + (\beta^2 - 1) \sin \theta) \sin \theta + \alpha_0^2.$$

where $\beta^2 = \frac{\alpha_1^2 - \alpha_0^2}{\alpha_2^2 - \alpha_0^2}$. Finally, let $g_+(\xi, \theta; \alpha) = \max\{0, g(\xi, \theta; \alpha)\}$.

Theorem 1.1. *Let g_+ be defined as above. Then, we have that*

$$w\text{-}\lim_{k \rightarrow \infty} d\rho_{DS}(x, k; \alpha) = \frac{1}{\pi} \int_0^\pi \int_0^{\pi/2} F(x; \theta, \xi, \alpha) \cos \theta \, d\xi d\theta$$

where F is a convex combination of delta functions given by

$$F(x; \theta, \xi, \alpha) = \frac{1}{4} \delta \left(x - \frac{1}{2} \sqrt{g_+(\xi, \theta; \alpha)} \right) + \frac{3}{4} \delta \left(x - \frac{3}{2} \sqrt{g_+(\xi, \theta; \alpha)} \right).$$

The weak limit is taken with respect to $C_c(\mathbb{R}^+)$.

The proof of Theorem 1.1 is given in the third section of the paper. In the second section, we show how one can separate the variables for the eigenvalue problem $-L_\alpha \psi = \lambda \psi$ and its connection to the Lamé equation. In particular, we will show how the spectrum of the operators $-L_\alpha$ can be explicitly computed through the Lamé equation.

2 Separation of variables and the Lamé equation

As we mentioned earlier, $-\Delta_{S^2}$ and $-L_\alpha$ are commuting, self-adjoint, elliptic operators on $L^2(S^2)$, hence they possess a Hilbert basis of joint eigenfunctions that form a class of spherical harmonics. Rather than working with the standard spherical harmonics Y_k^m , we introduce a more suitable class of spherical harmonics for our purpose, the so-called *Lamé harmonics* [BT, WW].

In terms of the Euclidean coordinates $(x, y, z) \in \mathbb{R}^3$, the Lamé harmonics of degree k are written as

$$\psi(x, y, z) = x^{\gamma_1} y^{\gamma_2} z^{\gamma_3} \prod_{j=0}^{\frac{1}{2}(k-|\gamma|)} \left(\frac{x^2}{\theta_j - \alpha_0^2} + \frac{y^2}{\theta_j - \alpha_1^2} + \frac{z^2}{\theta_j - \alpha_2^2} \right) \quad (2.1)$$

where $\gamma_i \in \{0, 1\}$ and $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$; the value of $|\gamma|$ is chosen so that $k - \gamma$ is even. The values of the parameters θ_j are determined by the condition $\Delta_{\mathbb{R}^3} \psi = 0$. A simple computation shows that the θ_j 's must satisfy Niven's equation

$$\sum_{j=0}^2 \frac{\gamma_j}{\theta_i - \alpha_j} + \sum_{j \neq i} \frac{1}{\theta_i - \theta_j} = 0, \quad (i = 1, \dots, \frac{1}{2}(k - |\gamma|)).$$

Based on Whittaker-Watson [WW] terminology, we say that ψ is of the first, second, third or fourth species if $|\gamma| = 0$, $|\gamma| = 1$, $|\gamma| = 2$ or $|\gamma| = 3$ respectively. Note that there is no Lamé harmonics of the second and fourth species for k even, whereas for k odd, there is none of the first and third species. We will see later on that there exists respectively $k/2 + 1$, $3(k+1)/2$, $3k/2$ and $(k-1)/2$ linearly independent Lamé harmonics of the first, second, third and fourth species. In particular, for any positive integer k , there exist $2k + 1$ linearly independent Lamé harmonics, hence they form a basis for the space of spherical harmonics.

2.1 Sphero-Conal coordinates

In order to describe the Lamé harmonics in greater detail, it is useful to introduce a different system of coordinates on S^2 , namely the sphero-conal coordinates [Sp, Vo]. We denote these by (u_1, u_2) . They are defined for any given positive real constants $\alpha_0^2 < \alpha_1^2 < \alpha_2^2$ by the zeros of the rational function

$$R(u) = \frac{x^2}{u - \alpha_0^2} + \frac{y^2}{u - \alpha_1^2} + \frac{z^2}{u - \alpha_2^2}$$

where $(x, y, z) \in \mathbb{R}^3$. From the graph of $R(u)$, it is easy to see that $\alpha_0^2 < u_1 < \alpha_1^2 < u_2 < \alpha_2^2$.

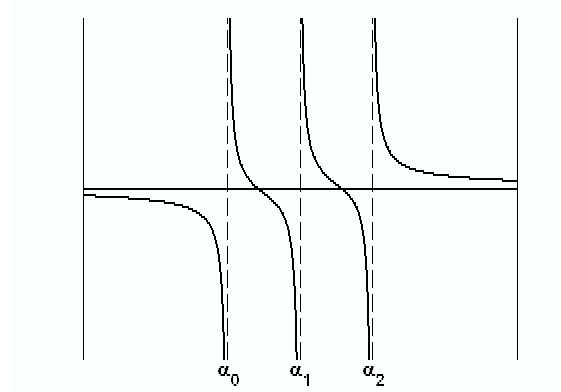


Figure 1: The graph of $R(u)$ for fixed values of x, y, z and α_i . The α_i correspond to the vertical asymptotes. The intersections with the u -axis are the two roots of $R(u)$ corresponding to the values of u_i .

The equation $R(u) = 0$ is invariant under rescaling $(x, y, z) \mapsto (tx, ty, tz)$, so the coordinates (u_1, u_2) are indeed coordinates on S^2 under the assumption $x^2 + y^2 + z^2 = 1$. They take their name from the fact that they can be obtained by the intersection of the unit sphere with confocal cones.

The relations between the sphero-conal and Euclidean coordinates are given by

$$\begin{aligned} x^2 &= \frac{(u_1 - \alpha_0^2)(u_2 - \alpha_0^2)}{(\alpha_2^2 - \alpha_0^2)(\alpha_1^2 - \alpha_0^2)}, \\ y^2 &= \frac{(u_1 - \alpha_1^2)(u_2 - \alpha_1^2)}{(\alpha_2^2 - \alpha_1^2)(\alpha_0^2 - \alpha_1^2)}, \\ z^2 &= \frac{(u_1 - \alpha_2^2)(u_2 - \alpha_2^2)}{(\alpha_0^2 - \alpha_2^2)(\alpha_1^2 - \alpha_2^2)}. \end{aligned}$$

In particular, (u_1, u_2) form an orthogonal system of coordinates on S^2 . This can easily be seen by considering the vectors $\vec{r}_i = (\partial_{u_i} x, \partial_{u_i} y, \partial_{u_i} z)$ for which

$$\begin{aligned} \vec{r}_1 \cdot \vec{r}_2 &= \frac{x^2}{(u_1 - \alpha_0^2)(u_2 - \alpha_0^2)} + \frac{y^2}{(u_1 - \alpha_1^2)(u_2 - \alpha_1^2)} + \frac{z^2}{(u_1 - \alpha_2^2)(u_2 - \alpha_2^2)} \\ &= \frac{R(u_1) - R(u_2)}{u_2 - u_1} \\ &= 0. \end{aligned}$$

2.2 Separation of variables

The great advantage of sphero-conal coordinates over other coordinate systems on S^2 is that they allow us to simultaneously separate variables in both of the spectral problems for $-\Delta_{S^2}$ and $-L_\alpha$ (see [Sp]). For example, in these coordinates, the Laplace equation $-\Delta_{S^2} \psi = k(k+1)\psi$ takes the form

$$\frac{4}{u_2 - u_1} \sum_{i=1}^2 (-1)^i \left[\sqrt{A(u_i)} \frac{\partial}{\partial u_i} \left(\sqrt{A(u_i)} \frac{\partial \psi}{\partial u_i} \right) \right] = k(k+1)\psi \quad (2.2)$$

where $A(u_i) = (u_i - \alpha_0^2)(u_i - \alpha_1^2)(u_i - \alpha_2^2)$. One can then separate the variables and write $\psi(u_1, u_2) = \psi_1(u_1)\psi_2(u_2)$. Denoting the separation constant by $-\lambda$, it follows directly from (2.2) that both ψ_1 and ψ_2 are solutions of the same *Lamé equation*

$$A(x)\psi_i''(x) + \frac{1}{2}A'(x)\psi_i'(x) = \frac{1}{4}(k(k+1)x - \lambda)\psi_i(x) \quad (i = 1, 2). \quad (2.3)$$

From the general theory of Lamé equation [WW], it is well known that the solutions of (2.3) are given by the Lamé functions

$$\psi_1(x) = \psi_2(x) = |x - \alpha_0^2|^{\gamma_1/2} |x - \alpha_1^2|^{\gamma_2/2} |x - \alpha_2^2|^{\gamma_3/2} \phi(x) \quad (2.4)$$

where ϕ is a polynomial of degree $(k - |\gamma|)/2$ with γ chosen as above. Consequently, the joint eigenfunctions of $-\Delta_{S^2}$ and $-L_\alpha$ are given by

$$\psi(u_1, u_2) = \prod_{j=1}^2 |u_j - \alpha_0^2|^{\gamma_1/2} |u_j - \alpha_1^2|^{\gamma_2/2} |u_j - \alpha_2^2|^{\gamma_3/2} \phi(u_j) \quad (2.5)$$

Note that, up to a constant depending only on the α 's and the solutions θ_j 's of the Niven's equations, (2.5) are the Lamé harmonics (2.1) expressed in sphero-conal coordinates.

Based on these observations, we can now compute the eigenvalues of $-L_\alpha$. Let E be such an eigenvalue; we will show that $E = \lambda$, the separation constant obtained previously. First, we use the fact that

$$-(L_x^2 + L_y^2 + L_z^2)\psi = -\Delta_{S^2}\psi = k(k+1)\psi$$

to deduce that

$$(\alpha_0^2 - \alpha_1^2)L_x^2\psi + (\alpha_2^2 - \alpha_1^2)L_z^2\psi = (\alpha_1^2k(k+1) - E)\psi.$$

In terms of sphero-conal coordinates, we can rewrite last equation as

$$\begin{aligned} \frac{4}{u_2 - u_1} \left[(\alpha_1^2 + u_2)\sqrt{A(u_1)}\frac{\partial}{\partial u_1} \left(\sqrt{A(u_1)}\frac{\partial \psi}{\partial u_1} \right) \right. \\ \left. - (\alpha_1^2 + u_1)\sqrt{A(u_2)}\frac{\partial}{\partial u_2} \left(\sqrt{A(u_2)}\frac{\partial \psi}{\partial u_2} \right) \right] = (\alpha_1^2k(k+1) - E)\psi. \end{aligned}$$

Upon separating the variables, $\psi(u_1, u_2) = \psi_1(u_1)\psi_2(u_2)$, we obtain

$$A(u_i)\psi_i''(u_i) + \frac{1}{2}A'(u_i)\psi_i'(u_i) = \frac{1}{4}(\mu u_i - E)\psi_i(u_i) \quad (i = 1, 2). \quad (2.6)$$

By comparison of (2.6) with (2.3), we conclude that $\mu = k(k+1)$ and $E = \lambda$ as desired. All that remains to prove is that we get all the possible eigenvalues of $-L_\alpha$ in this way. This is a consequence of the following result due to Stieltjes and Szëgo (see [Sz], §6.3):

Theorem 2.1. *Let ρ_0, ρ_1, ρ_2 be any three real positive numbers and let a_1, a_2, a_3 be any three real distinct numbers. There exist exactly $m+1$ distinct real numbers ν for which the generalized Lamé equation*

$$A(x)y''(x) + \sum_{j=0}^2 \rho_j \prod_{i \neq j} (x - a_i)y'(x) = (m(m+1 + |\rho|)x - \nu)y(x) \quad (2.7)$$

has a polynomial solution y of degree m . Moreover, the $m+1$ polynomial solutions obtained in this way are linearly independent.

Replacing the expression of the Lamé function ψ_i given in (2.4) into (2.3), one can easily verify that the polynomial ϕ of degree $(k - |\gamma|)/2$ satisfies the generalized Lamé equation

$$\begin{aligned} A(x)\phi''(x) + \sum_{j=0}^2 \left(\gamma_j + \frac{1}{2} \right) \prod_{l \neq j} (x - \alpha_l^2)\phi'(x) \\ = \frac{1}{4} \left((k - |\gamma|)(k + |\gamma| + 1)x - \lambda + D(\alpha, \gamma) \right) \phi(x), \end{aligned} \quad (2.8)$$

where $D(\alpha, \gamma) = (\alpha_0^2 + \alpha_1^2)\gamma_2 + (\alpha_0^2 + \alpha_2^2)\gamma_1 + (\alpha_1^2 + \alpha_2^2)\gamma_0 + 2\gamma_0\gamma_1\alpha_2^2 + 2\gamma_1\gamma_2\alpha_0^2 + 2\gamma_0\gamma_2\alpha_1^2$. The values taken by $\nu = \lambda - D(\alpha, \gamma)$ in terms of the different values of γ are given in the table below.

species	$\gamma_0, \gamma_1, \gamma_2$	ν
1	$\gamma_0 = \gamma_1 = \gamma_2 = 0$	λ
2	$\gamma_0 = 1, \gamma_1 = \gamma_2 = 0$	$\lambda - \alpha_1^2 - \alpha_2^2$
	$\gamma_1 = 1, \gamma_0 = \gamma_2 = 0$	$\lambda - \alpha_0^2 - \alpha_2^2$
	$\gamma_2 = 1, \gamma_0 = \gamma_1 = 0$	$\lambda - \alpha_0^2 - \alpha_1^2$
3	$\gamma_0 = 0, \gamma_1 = \gamma_2 = 1$	$\lambda - 4\alpha_0^2 - \alpha_1^2 - \alpha_2^2$
	$\gamma_1 = 0, \gamma_0 = \gamma_2 = 1$	$\lambda - \alpha_0^2 - 4\alpha_1^2 - \alpha_2^2$
	$\gamma_2 = 0, \gamma_0 = \gamma_1 = 1$	$\lambda - \alpha_0^2 - \alpha_1^2 - 4\alpha_2^2$
4	$\gamma_0 = \gamma_1 = \gamma_2 = 1$	$\lambda - 4(\alpha_0^2 + \alpha_1^2 + \alpha_2^2)$

Table 1: The values taken by ν

By Stieltjes' result with $\rho_i = \gamma_i + 1/2$, we deduce that there are exactly $(k - |\gamma|)/2 + 1$ distinct value ν for which (2.8) has a polynomial solution ϕ of degree $(k - |\gamma|)/2$. In particular, the number of Lamé harmonics of degree k and of specie 1 is $k/2 + 1$, of species 2 is $3(k + 1)/2$, of specie 3 is $3k/2$ and of specie 4 is $(k - 1)/2$. It follows that for any $k \in \mathbb{N}$, there exist $2k + 1$ linearly independent Lamé harmonics, so they form a Hilbert basis of $L^2(S^2)$.

Furthermore, for each $k \in \mathbb{N}$, we also obtain $2k + 1$ values of ν (multiplicity included) to which correspond by Table 1, $2k + 1$ values of λ . In other words, the eigenvalues of the linearly independent Lamé harmonics of degree k are exactly given by the $2k + 1$ values of λ . Therefore, we have shown the first part of the following theorem.

Theorem 2.2. *Let $\alpha = (\alpha_0^2, \alpha_1^2, \alpha_2^2) \in \Lambda^3$, then the spectrum of the operator $-L_\alpha$ is given by all numbers λ appearing on the RHS of the Lamé equation (2.3). Moreover, the λ 's corresponding to the Lamé harmonics of degree k lie within the interval $(\alpha_0^2(k - 3)(k + 1), \alpha_2^2k(k + 4) + 4|\alpha|)$.*

The second part is an immediate consequence of a result due to Van Vleck [Va] where he proves that all numbers ν corresponding to the polynomial solutions of degree m of the generalized Lamé equation (2.7) lie inside the interval $(\alpha_0^2m(m + 1 + |\rho|), \alpha_2^2m(m + 1 + |\rho|))$. It follows from this and (2.8) that the eigenvalues λ lie inside the interval

$$\min_{\gamma} \{ \alpha_0^2(k - |\gamma|)(k + |\gamma| + 1) + D(\alpha, \gamma) \} \leq \lambda \leq \max_{\gamma} \{ \alpha_2^2(k - |\gamma|)(k + |\gamma| + 1) + D(\alpha, \gamma) \}.$$

Since $\gamma_i \in \{0, 1\}$, it is then easy to see that

$$\min_{\gamma} \{ \alpha_0^2(k - |\gamma|)(k + |\gamma| + 1) + D(\alpha, \gamma) \} \geq \alpha_0^2(k - 3)(k + 1)$$

and

$$\max_{\gamma} \{\alpha_2^2(k - |\gamma|)(k + |\gamma| + 1) + D(\alpha, \gamma)\} \leq \alpha_2^2 k(k + 4) + 4|\alpha|$$

from which the conclusion of the theorem follows.

3 Proof of Theorem 1.1

Based on the different species of the eigenvalues, we partition the spectrum of $-L_\alpha$ into four disjoint subsets $\sigma_1^k, \dots, \sigma_4^k$ defined by

$$\sigma_i^k := \{\lambda : \lambda \text{ is an eigenvalue of a Lamé harmonics of degree } k \text{ and of species } i\}.$$

For each $k \in \mathbb{N}$, we denote the eigenvalues of $\sqrt{-L_\alpha}$ corresponding to the $2k + 1$ Lamé harmonics of degree k by

$$\sqrt{\lambda_{-k}^k(\alpha)} < \sqrt{\lambda_{-k+1}^k(\alpha)} < \dots < \sqrt{\lambda_k^k(\alpha)}.$$

Based on the definition of the σ_i , we can decompose $d\rho_{DS}(\varphi; k, \alpha)$ into four disjoint sums, i.e.

$$\begin{aligned} d\rho_{DS}(\varphi; k, \alpha) &= \frac{1}{2k+1} \sum_{j=-k}^k \varphi \left(\frac{\sqrt{\lambda_j^k(\alpha)}}{k} \right) + \mathcal{O} \left(\frac{1}{k} \right) \\ &= \frac{1}{2k+1} \sum_{i=1}^4 \sum_{\lambda \in \sigma_i^k} \varphi \left(\frac{\sqrt{\lambda}}{k} \right) + \mathcal{O} \left(\frac{1}{k} \right). \end{aligned} \quad (3.1)$$

As we mentioned before, when k is even, only the Lamé harmonics of the first and third species will contribute to the sum above, whereas only the second and fourth species will contribute when k is odd. Therefore, we can write

$$\sum_{j=-k}^k \varphi \left(\frac{\sqrt{\lambda_j^k}}{k} \right) = \begin{cases} \sum_{\lambda \in \sigma_1^k} \varphi \left(\frac{\sqrt{\lambda}}{k} \right) + \sum_{\lambda \in \sigma_3^k} \varphi \left(\frac{\sqrt{\lambda}}{k} \right), & k \text{ even} \\ \sum_{\lambda \in \sigma_2^k} \varphi \left(\frac{\sqrt{\lambda}}{k} \right) + \sum_{\lambda \in \sigma_4^k} \varphi \left(\frac{\sqrt{\lambda}}{k} \right), & k \text{ odd.} \end{cases}$$

The key observation here is that the eigenvalues can be obtained by simply regarding the polynomial solution of the generalized Lamé equation (2.8). More precisely, we introduce the sets Z_i , $i = 1, 2, 3, 4$, defined by

$$Z_i^k := \{\nu \mid \text{There exist } \lambda \in \sigma_i^k \text{ and } \gamma \in \{0, 1\}^3 \text{ such that } \nu = \lambda - D(\alpha, \gamma)\}.$$

Consequently, the four sums above can now be taken over the sets Z_i^k instead of σ_i^k . That is,

$$\sum_{\lambda \in \sigma_i^k} \varphi \left(\frac{\sqrt{\lambda}}{k} \right) = \sum_{\nu \in Z_i^k} \varphi \left(\frac{\sqrt{\nu + D(\alpha, \gamma)}}{k} \right) \quad (3.2)$$

Moreover, since φ is compactly supported, we can approximate uniformly φ by smooth functions. Without loss of generality, we may therefore assume that φ satisfies

$$\varphi\left(\frac{\sqrt{\nu + D(\alpha, \gamma)}}{k}\right) = \varphi\left(\frac{\sqrt{\nu}}{k}\right) + \mathcal{O}\left(\frac{1}{k}\right)$$

since $D(\alpha, \gamma) = \mathcal{O}(1)$. The equation (3.2) easily implies that

$$\frac{1}{|\sigma_i^k|} \sum_{\lambda \in \sigma_i^k} \varphi\left(\frac{\sqrt{\lambda}}{k}\right) = \frac{1}{|Z_i^k|} \sum_{\nu \in Z_i^k} \varphi\left(\frac{\sqrt{\nu}}{k}\right) + \mathcal{O}\left(\frac{1}{k}\right). \quad (3.3)$$

The asymptotic of the sums in RHS of (3.3) are obtained through the following lemma.

Lemma 3.1. *Let ν_0, \dots, ν_m denote the $m+1$ real numbers for which the Lamé equation*

$$A(x)y''(x) + \sum_{j=0}^2 \rho_j \prod_{i \neq j} (x - \alpha_i^2) y'(x) = (m(m+1 + |\rho|)x - \nu)y(x)$$

admits a polynomial solution y of degree m . For any $\varphi \in C_c(\mathbb{R}^+)$, we have that

$$\frac{1}{m+1} \sum_{j=0}^m \varphi\left(\frac{\sqrt{\nu_j}}{m}\right) = \frac{1}{\pi} \int_0^\pi \int_0^{\pi/2} \varphi\left(\sqrt{g_+(\xi, \theta; \alpha)}\right) \cos \theta \, d\theta \, d\xi + \mathcal{O}\left(\frac{1}{m}\right)$$

where $g_+(\xi, \theta; \alpha) = \max\{0, (\alpha_1^2 - \alpha_0^2)(\beta \sin \theta \cos \xi + (\beta^2 - 1) \sin^2 \theta) + \alpha_0^2\}$, and $\beta^2 = \frac{\alpha_2^2 - \alpha_1^2}{\alpha_1^2 - \alpha_0^2}$.

The proof of Lemma 3.1 is rather long and technical, so we prefer to postpone it until the end of the present section. With this lemma in hand, we can now complete the proof of Theorem 1.1. As a consequence of Lemma 3.1, we obtain for k even,

$$\begin{aligned} \frac{1}{2k+1} \sum_{j=-k}^k \varphi\left(\frac{\sqrt{\lambda_j^k(\alpha)}}{k}\right) \\ &= \frac{1}{2k+1} \left[\sum_{\lambda \in \sigma_1^k} \varphi\left(\frac{\sqrt{\lambda}}{k}\right) + \sum_{\lambda \in \sigma_3^k} \varphi\left(\frac{\sqrt{\lambda}}{k}\right) \right] \\ &= \frac{1}{2k+1} \left[\sum_{\nu \in Z_1^k} \varphi\left(\frac{\sqrt{\nu}}{k}\right) + \sum_{\nu \in Z_3^k} \varphi\left(\frac{\sqrt{\nu}}{k}\right) \right] + \mathcal{O}\left(\frac{1}{k}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \left[\frac{1}{k/2+1} \sum_{\nu \in Z_1^k} \varphi \left(\frac{1}{2} \frac{\sqrt{\nu}}{k/2} \right) \right] \\
&+ \frac{3}{4} \left[\frac{1}{3k/2} \sum_{\nu \in Z_3^k} \varphi \left(\frac{3}{2} \frac{\sqrt{\nu}}{3k/2} \right) \right] + \mathcal{O} \left(\frac{1}{k} \right)
\end{aligned} \tag{3.4}$$

By Lemma 3.1, the first sum in the brackets of (3.4) is equal to

$$\frac{1}{\pi} \int_0^\pi \int_0^{\pi/2} \varphi \left(\frac{1}{2} g_+(\xi, \theta; \alpha) \right) \cos \theta \, d\theta \, d\xi + \mathcal{O} \left(\frac{1}{k} \right), \tag{3.5}$$

and the second sum in brackets of (3.4) is equal to

$$\frac{1}{\pi} \int_0^\pi \int_0^{\pi/2} \varphi \left(\frac{3}{2} g_+(\xi, \theta; \alpha) \right) \cos \theta \, d\theta \, d\xi + \mathcal{O} \left(\frac{1}{k} \right), \tag{3.6}$$

Combining equations (3.5) and (3.6), we deduce that

$$\begin{aligned}
&\frac{1}{2k+1} \sum_{j=-k}^k \varphi \left(\frac{\sqrt{\lambda_j^k(\alpha)}}{k} \right) \\
&= \frac{1}{\pi} \int_0^\pi \int_0^{\pi/2} F(\varphi; \xi, \theta; \alpha) \cos \theta \, d\theta \, d\xi + \mathcal{O} \left(\frac{1}{k} \right)
\end{aligned} \tag{3.7}$$

where the function F is defined by

$$F(\varphi; \xi, \theta; \alpha) = \frac{1}{4} \varphi \left(\frac{1}{2} g(\xi, \theta; \alpha) \right) + \frac{3}{4} \varphi \left(\frac{3}{2} g(\xi, \theta; \alpha) \right).$$

The conclusion of Theorem 1.1 for k even then follows from (3.1) and (3.7). Similarly, for k odd, we have that

$$\begin{aligned}
&\frac{1}{2k+1} \sum_{j=-k}^k \varphi \left(\frac{\sqrt{\lambda_j^k(\alpha)}}{k} \right) \\
&= \frac{1}{2k+1} \left[\sum_{\lambda \in \sigma_2^k} \varphi \left(\frac{\sqrt{\lambda}}{k} \right) + \sum_{\lambda \in \sigma_4^k} \varphi \left(\frac{\sqrt{\lambda}}{k} \right) \right] \\
&= \frac{1}{2k+1} \left[\sum_{\nu \in Z_2^k} \varphi \left(\frac{\sqrt{\nu}}{k} \right) + \sum_{\nu \in Z_4^k} \varphi \left(\frac{\sqrt{\nu}}{k} \right) \right] + \mathcal{O} \left(\frac{1}{k} \right) \\
&= \frac{3}{4} \left[\frac{1}{3k/2} \sum_{\nu \in Z_2^k} \varphi \left(\frac{3}{2} \frac{\sqrt{\nu}}{3k/2} \right) \right] \\
&+ \frac{1}{4} \left[\frac{1}{(k-1)/2} \sum_{\nu \in Z_4^k} \varphi \left(\frac{1}{2} \frac{\sqrt{\nu}}{k/2} \right) \right] + \mathcal{O} \left(\frac{1}{k} \right).
\end{aligned} \tag{3.8}$$

As for the case k even, we apply Lemma 3.1 to conclude that (3.7) holds when k is a positive odd integer.

To complete the proof of Theorem 1.1, it remains to prove Lemma 3.1.

3.1 Proof of Lemma 3.1

According to Theorem 2.1 with $a_0 = -1$, $a_0 = 0$ and $a_2 = \beta^2 > 0$, there exist $m + 1$ real values $\tilde{\nu}_0, \dots, \tilde{\nu}_m$ for which the generalized Lamé equation

$$x(x - \beta^2)(x + 1)Y''(x) + [\rho_0 x(x - \beta^2) + \rho_1(x + 1)(x - \beta^2) + \rho_2 x(x + 1)]Y'(x) = (m(m + 1 + |\rho|)x - \tilde{\nu})Y(x), \quad (3.9)$$

admits a polynomial solution Y of degree m . First, we show that for any $\varphi \in C_c(\mathbb{R}^+)$

$$\frac{1}{m+1} \sum_{j=0}^m \varphi\left(\frac{\tilde{\nu}_j}{m^2}\right) = \frac{1}{\pi} \int_0^\pi \int_0^{\pi/2} \varphi(h(\xi, \theta; \alpha)) \cos \theta \, d\theta \, d\xi + \mathcal{O}\left(\frac{1}{m}\right) \quad (3.10)$$

where $h(\xi, \theta; \alpha) = \beta \sin \theta \cos \xi + (\beta^2 - 1) \sin^2 \theta$.

The starting point in proving (3.10) consists of establishing a three-term recurrence relation satisfied by the Lamé polynomials Y . In particular, this will allow us to obtain the eigenvalues of $-L_\alpha$ as the those of some tridiagonal matrix.

More precisely, we consider a Lamé polynomial of degree m of the form

$$Y(x) = \sum_{j=0}^m a_j x^j.$$

If we replace the expression for $Y(x)$ into the Lamé equation (3.9), we obtain the following three-term recurrence relation:

$$A_j(\rho, \beta)a_j + B_j(\rho, \beta)a_{j+1} + C_j(\rho, \beta)a_{j-1} = \tilde{\nu}a_j \quad (j = 0, \dots, m) \quad (3.11)$$

where $a_{-1} = 0$, $a_{m+1} = 0$, and

$$\begin{cases} A_j(\rho, \beta) = (\beta^2 - 1)j(j - 1 + \rho_1) - \rho_2 j + \beta^2 \rho_0 j, \\ B_j(\rho, \beta) = (j + 1)(j + \rho_1)\beta^2 \\ C_j(\rho, \beta) = \mu - (j - 1)(j - 2 + |\rho|). \end{cases} \quad (3.12)$$

Note that, as a result of the above, $A_0 = B_m = C_0$. These relations are more conveniently expressed in matrix form. Indeed, if we introduce the tridiagonal matrix $A = (a_{ij})$, $i, j = 0, \dots, m$, given by

$$a_{ij} = \begin{cases} \frac{B_i(\rho, \beta)}{\mu} & \text{if } i = j - 1 \\ \frac{A_i(\rho, \beta)}{\mu} & \text{if } i = j \\ \frac{C_i(\rho, \beta)}{\mu} & \text{if } i = j + 1, \end{cases} \quad (3.13)$$

then the three-term recurrence relation (3.11) implies that

$$AX = \frac{\tilde{\nu}}{\mu}X,$$

where $X = (a_0, a_1, \dots, a_m)^T$. Throughout the rest of the proof, we denote by $\frac{\tilde{\nu}_0}{\mu}, \dots, \frac{\tilde{\nu}_m}{\mu}$ the $m+1$ eigenvalues of A . Note that the components of the eigenvectors X are exactly the coefficients of the Lamé polynomials Y .

We will divide the rest of the proof into several lemmas. The first one consists of computing the trace of the powers A^n for any $n \in \mathbb{N}$.

Lemma 3.2. *We have that*

$$\begin{aligned} \text{Tr}(A^n) &= \sum_{j=0}^{\llbracket n/2 \rrbracket} \binom{n}{j, j, n-2j} \\ &\quad \times \sum_{i=1}^m \left(1 - \frac{i^2}{m^2}\right)^j \left(\frac{i^2}{m^2}\right)^{n-j} (\beta^2 - 1)^{n-2j} \beta^{2j} + \mathcal{O}(1) \end{aligned} \quad (3.14)$$

for any positive integer n . Here, $\llbracket n/2 \rrbracket$ denotes the greatest integer less or equal to $n/2$.

Proof of Lemma 3.2: We decompose A as a sum of three matrices, $A = L + D + U$, where $D = \frac{1}{\mu} \text{diag}(0, A_1, \dots, A_m)$ and

$$L = \frac{1}{\mu} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ C_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & C_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_m & 0 \end{pmatrix}, \quad U = \frac{1}{\mu} \begin{pmatrix} 0 & B_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & B_{m-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The trace of A^n is then given by the trace of $(L + D + U)^n$. When we expand last expression, the non-commutativity of the matrices L, D and U implies that the trace of A^n is the sum of 3^n terms of the form

$$M_1 M_2 \cdots M_n,$$

where $M_i = L, D$, or U . This is unmanageable in its full generality for arbitrary n . However, we are interested primarily in the asymptotic information contained in the trace, which allows us to make significant simplifications.

First, we point out that our need to consider A^n stems from the fact that we will use polynomials to approximate the continuous function φ in Lemma 3.1. Thus, we need to extract asymptotic information about $\text{Tr}(A^n)$ for fixed, but arbitrary n . In our case, we will ultimately be taking a limit $m \rightarrow \infty$ for fixed n , and so in this limit, $n/m \rightarrow 0$.

Secondly, we exploit the fact that the terms $M_1 M_2 \cdots M_n$ are products of matrices, each being lower diagonal (L), diagonal (D), or upper diagonal (U). This allows us to make definite statements about the *zero structure* of the matrix products, i.e., the entries that are necessarily zero in the matrix product. For example, multiplication on the left or right by a diagonal matrix preserves the zero structure: LD and DL are both lower diagonal if L is. The analogous statement holds for UD and DU . The effect of multiplying by L or U is only slightly less simple. In fact, as far as the effect on zero structure is concerned, L and U behave like quantum mechanical creation and annihilation operators (respectively). In detail, if we denote by R_M (resp. L_M) the operation of right (resp. left) multiplication by a matrix M , then for any matrix B :

- (i) $R_U B$ corresponds to shifting all columns of B one place to the right: $\text{col}_{i+1}(R_U B) = \text{col}_i(B)$, creating a zero column in the first column.
- (ii) $R_L B$ corresponds to shifting all columns of B one place to the left: $\text{col}_{i-1}(R_L B) = \text{col}_i(B)$, creating a zero column in the last column.
- (iii) $L_U B$ corresponds to shifting all rows of B up one place: $\text{row}_{i-1}(L_U B) = \text{row}_i(B)$, creating a zero row in the last row.
- (iv) $L_L B$ corresponds to shifting all rows of B down one place: $\text{row}_{i+1}(L_L B) = \text{row}_i(B)$, creating a zero row in the first row.

As a result, the diagonal of a term $M_1 M_2 \cdots M_n$ in A^n will have zero trace unless the number of factors j of L is the same as the number of factors of U . The remaining $n - 2j$ factors must all be D . Thus, many of the 3^n terms do not contribute to $\text{Tr}(A^n)$.

The last issue concerns the lack of commutativity in the terms that do contribute to the trace. Some of these terms are of the form

$$(LU)^j D^{n-2j}, \quad j = 0, \dots, \lfloor n/2 \rfloor. \quad (3.15)$$

Since LU and D are diagonal, the trace is particularly simple to compute in the case of the *canonical terms* (3.15):

$$\begin{aligned} \text{Tr}(M_1 M_2 \cdots M_n) &= \text{Tr}((LU)^j D^{n-2j}) \\ &= \sum_{i=1}^m \left(1 - \frac{i^2}{m^2}\right)^j \left(\frac{i^2}{m^2}\right)^{n-j} (\beta^2 - 1)^{n-2j} \beta^{2j} + \mathcal{O}(1). \end{aligned}$$

Noncanonical terms will differ from canonical terms only at order $\mathcal{O}(n/m) = \mathcal{O}(1/m)$, and so for asymptotic purposes, we may assume that all terms have the canonical form (3.15). To see this, note that multiplication of matrices of the form L, D , and U constitutes a shifting of their rows and columns. For terms with n factors, the number of shifts is at most n . Being products of matrices that

are (lower, upper) diagonal, the noncanonical terms will yield sums of products of the form

$$\Gamma_p \Delta_q,$$

where $\Gamma_p, \Delta_q \in \{A_l/\mu, B_l/\mu, C_l/\mu \mid l = 0, 1, \dots, m\}$ and $|p - q| = \mathcal{O}(n)$. As an example,

$$\begin{aligned} \frac{A_p}{\mu} \frac{B_q}{\mu} &= \beta^2(\beta^2 - 1) \frac{p^2 q^2}{m^4} + \mathcal{O}\left(\frac{1}{m}\right) \\ &= \beta^2(\beta^2 - 1) \frac{p^2(p + \mathcal{O}(n))^2}{m^4} + \mathcal{O}\left(\frac{1}{m}\right) \\ &= \beta^2(\beta^2 - 1) \frac{p^4}{m^4} + \mathcal{O}(n/m) + \mathcal{O}\left(\frac{1}{m}\right) \\ &= \beta^2(\beta^2 - 1) \frac{p^4}{m^4} + \mathcal{O}\left(\frac{1}{m}\right). \end{aligned}$$

Since there are exactly $\binom{n}{j, j, n-2j}$ matrices $M_1 M_2 \cdots M_n$ that contain j factors of L , j factors of U and $(n - 2j)$ factors of D , we finally deduce that

$$\begin{aligned} \text{Tr}(A^n) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j, j, n-2j} \text{Tr}((LU)^j D^{n-2j}) + \mathcal{O}(1) \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j, j, n-2j} \sum_{i=1}^m \left(1 - \frac{i^2}{m^2}\right)^j \left(\frac{i^2}{m^2}\right)^{n-j} (\beta^2 - 1)^{n-2j} \beta^{2j} \\ &\quad + \mathcal{O}(1). \end{aligned} \tag{3.16}$$

This completes the proof of the Lemma. \square

The next result deals with the inner sum $\sum_{i=1}^m \left(1 - \frac{i^2}{m^2}\right)^j \left(\frac{i^2}{m^2}\right)^{n-j}$ in (3.16). As the next lemma shows, this sum is asymptotically given by a Beta integral.

Lemma 3.3. *We have that*

$$\frac{1}{m} \sum_{i=0}^m \left(1 - \frac{i^2}{m^2}\right)^j \left(\frac{i^2}{m^2}\right)^{n-j} = \frac{1}{2} \mathbf{B}(j+1, n-j+1/2) + \mathcal{O}\left(\frac{1}{m}\right) \tag{3.17}$$

where $\mathbf{B}(p, q)$ is the standard Beta integral defined by

$$\mathbf{B}(p, q) = 2 \int_0^{\pi/2} \cos^{2p-1} \theta \sin^{2q-1} \theta \, d\theta.$$

Proof of Lemma 3.3: This is obvious. The LHS of (3.17) is a Riemann sum for the function $(1 - x^2)^j (x^2)^{n-j}$ on $[0, 1]$, hence

$$\frac{1}{m} \sum_{i=0}^m \left(1 - \frac{i^2}{m^2}\right)^j \left(\frac{i^2}{m^2}\right)^{n-j} = \int_0^1 (1 - x^2)^j (x^2)^{n-j} \, dx + \mathcal{O}\left(\frac{1}{m}\right).$$

The conclusion of the lemma follows by making the substitution $x = \sin \theta$ and using the trigonometric representation of the Beta integral. \square

As a consequence of (3.16) and Lemma 3.2, it follows that

$$\begin{aligned} \frac{1}{m} \text{Tr}(A^n) &= \frac{1}{m} \sum_{i=0}^m \left(\frac{\nu_i}{\mu} \right)^n \\ &= \frac{1}{2} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j, j, n-2j} \mathbf{B}(j+1, n-j+1/2) (\beta^2 - 1)^{n-2j} \beta^{2j} \\ &\quad + \mathcal{O}\left(\frac{1}{m}\right). \end{aligned} \quad (3.18)$$

In order to evaluate the sum inside the integral sign, we use the *sinc* function defined by

$$\text{sinc}(x) = \begin{cases} 1 & \text{for } x = 0, \\ \frac{\sin x}{x} & \text{for } x \neq 0. \end{cases}$$

The key point here is to observe that $\text{sinc}(\pi x) = 0$ when x is a non-zero integer, and that $\text{sinc}(0) = 1$. Using this function, we can then replace the sum in (3.18) by the more appropriate sum over multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ such that $|\gamma| = n$. More precisely, we have

$$\begin{aligned} &\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j, j, n-2j} \mathbf{B}(j+1, n-j+1/2) (\beta^2 - 1)^{n-2j} \beta^{2j} \\ &= \sum_{|\gamma|=n} \binom{n}{\gamma} (\beta^2 - 1)^{\gamma_3} \beta^{\gamma_1 + \gamma_2} \mathbf{B}(\gamma) \text{sinc}(\pi(\gamma_1 - \gamma_2)). \end{aligned} \quad (3.19)$$

where $\mathbf{B}(\gamma) := \mathbf{B}\left(\frac{\gamma_1}{2} + \frac{\gamma_2}{2} + 1, n - \frac{\gamma_1}{2} - \frac{\gamma_2}{2} + \frac{1}{2}\right)$. Based on the representation of $\text{sinc}(x)$ as the integral

$$\text{sinc}(\pi(\gamma_1 - \gamma_2)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(\gamma_1 - \gamma_2)} d\xi, \quad (3.20)$$

the RHS of (3.19) can be written as

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{|\gamma|=n} \binom{n}{\gamma} (\beta^2 - 1)^{\gamma_3} \beta^{\gamma_1 + \gamma_2} \mathbf{B}(\gamma) e^{i\xi(\gamma_1 - \gamma_2)} d\xi. \quad (3.21)$$

Replacing $\mathbf{B}(\gamma)$ by the expression

$$\mathbf{B}(\gamma) = 2 \int_0^{\pi/2} (\cos \theta)^{\gamma_1 + \gamma_2 + 1} (\sin \theta)^{2n - \gamma_1 - \gamma_2} d\theta,$$

we can then use the Multinomial Theorem to evaluate the sum in (3.21). We obtain

$$\begin{aligned} \sum_{|\gamma|=n} \binom{n}{\gamma} (\beta^2 - 1)^{\gamma_3} \beta^{\gamma_1 + \gamma_2} (\cos \theta)^{\gamma_1 + \gamma_2} (\sin \theta)^{2n - \gamma_1 - \gamma_2} e^{i\xi(\gamma_1 - \gamma_2)} \\ = (\beta \cos \xi \sin 2\theta + (\beta^2 - 1) \sin^2 \theta)^n. \end{aligned} \quad (3.22)$$

If we denote by $h(\xi, \theta) = (\beta^2 \cos \xi \sin 2\theta + (\beta^2 - 1) \sin^2 \theta)$, then equations (3.18) through (3.22) imply that

$$\frac{1}{m} \text{Tr}(A^n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\pi/2} h^n(\xi, \theta) \cos \theta \, d\theta \, d\xi + \mathcal{O}\left(\frac{1}{m}\right).$$

The rest of proof of Lemma 3.1 follows by the standard functional calculus on the Banach algebra $M_m(\mathbb{R})$, the set of all matrices of order m with real entries. However, we can also complete the proof by simply observing that for any polynomial P ,

$$\frac{1}{m} \sum_{i=0}^m \text{Tr}(P(A)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\pi/2} P(h(\xi, \theta)) \cos \theta \, d\theta \, d\xi + \mathcal{O}\left(\frac{1}{m}\right). \quad (3.23)$$

Finally, Weierstrass' Theorem implies that for any compactly continuous function φ and any $\epsilon > 0$, there exists a polynomial P with

$$\sup_x |\varphi(x) - P(x)| < \epsilon/3. \quad (3.24)$$

This implies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\pi/2} |\phi(h(\xi, \theta)) - P(h(\xi, \theta))| \cos \theta \, d\theta \, d\xi < \epsilon/3. \quad (3.25)$$

From the Spectral Mapping Theorem and (3.24) we obtain

$$\left| \frac{1}{k} \text{Tr}(\varphi(A)) - \frac{1}{k} \text{Tr}(P(A)) \right| < \epsilon/3. \quad (3.26)$$

We choose m big enough in (3.23) so that

$$\left| \frac{1}{m} \sum_{i=0}^m \text{Tr}(P(A)) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\pi/2} P(h(\xi, \theta)) \cos \theta \, d\theta \, d\xi \right| < \epsilon/3. \quad (3.27)$$

As a consequence of (3.25) - (3.27)

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^m \text{Tr}(\varphi(A)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\pi/2} \varphi(h(\xi, \theta)) \cos \theta \, d\theta \, d\xi. \quad (3.28)$$

This completes the proof of (3.10) for $a_0 = -1$, $a_1 = 0$ and $a_2 = \beta^2$. In the case of interest to us, namely $a_0 = \alpha_0^2$, $a_1 = \alpha_1^2$ and $a_2 = \alpha_2^2$, we use the fact that the Lamé equation is invariant under affine transformations to make the change of variable $x \mapsto x(\alpha_1^2 - \alpha_0^2) + \alpha_0^2$. If we let y be the function defined by $y(x) := Y(x(\alpha_1^2 - \alpha_0^2) + \alpha_1^2)$, then it is not hard to show that y satisfies the standard Lamé equation

$$A(x)y''(x) + \sum_{j=0}^2 \rho_j \prod_{i \neq j} (x - \alpha_i^2) y'(x) = (\mu x - \nu)y(x)$$

where $\nu = \tilde{\nu}(\alpha_1^2 - \alpha_0^2) + \alpha_1^2 \mu$. From the fact that $-1 \leq \frac{\tilde{\nu}}{\mu} \leq \beta^2$, we easily deduce that $\alpha_0^2 \leq \frac{\nu}{\mu} \leq \alpha_2^2$.

Furthermore, if we introduce the function $\varphi_\alpha(x) := \varphi(x(\alpha_1^2 - \alpha_0^2) + \alpha_1^2)$ for any $\varphi \in C_c(\mathbb{R}^+)$, then we obtain that

$$\begin{aligned} \frac{1}{m} \sum_{i=0}^m \varphi\left(\frac{\nu_i}{m^2}\right) &= \frac{1}{m} \sum_{i=0}^m \varphi\left(\frac{\tilde{\nu}_i}{m^2}(\alpha_1^2 - \alpha_0^2) + \alpha_1^2\right) + \mathcal{O}\left(\frac{1}{m}\right) \\ &= \frac{1}{m} \sum_{i=0}^m \varphi_\alpha\left(\frac{\tilde{\nu}_i}{m^2}\right) + \mathcal{O}\left(\frac{1}{m}\right). \end{aligned} \quad (3.29)$$

It then follows by (3.28) and (3.29) that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^m \varphi\left(\frac{\nu_i}{m^2}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\pi/2} \varphi(g(\xi, \theta; \alpha)) \cos \theta \, d\theta \, d\xi \quad (3.30)$$

where $g(\xi, \theta; \alpha) = h(\xi, \theta; \alpha)(\alpha_1^2 - \alpha_0^2) + \alpha_1^2$. Since φ is supported in \mathbb{R}^+ , last equation remains valid if we replace $g(\xi, \theta; \alpha)$ by $g_+(\xi, \theta; \alpha) = \max\{0, g(\xi, \theta; \alpha)\}$ and $\varphi(x)$ by $\varphi(\sqrt{x})$. This completes the proof of Lemma 3.1. \square

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